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MECHANICS OF MULTIPLE PERIODIC BRITTLE MATRIX CRACKS IN UNIDIRECTIONAL FIBER-REINFORCED COMPOSITES

AUTAR K. KAW and GLEN H. BESTERFIELD
ENG 118, Mechanical Engineering Department, University of South Florida, Tampa, FL 33620, U.S.A.

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Abstract—A fracture model based on two-dimensional plane stress–strain elasticity theory is developed for the problem of periodic, interacting and regularly spaced matrix cracks in a unidirectional fiber-reinforced brittle matrix composite. The solution is obtained in terms of a hypersingular integral equation. The effects of the fiber reinforcement, and the spacing, the location and the length of cracks on the stress intensity factors at the crack tips and the maximum crack opening displacement in the composite are studied.

INTRODUCTION

Ceramic materials, such as glass, ceramic and silicon carbide, are being used for high temperature applications in many engineering components. The advantages in their use are their strength, low density, excellent corrosion and oxidation resistance, and low cost. Their main drawback, however, is that they are brittle in nature and therefore have a tendency to fail catastrophically. This has limited their use in relatively low stress applications, or where catastrophic failure is not a critical issue.

One of the most promising methods to increase the toughness of ceramics is by reinforcing them with continuous fibers such as silicon carbide and carbon. However, fiber-reinforced ceramic matrix composites are highly anisotropic and exhibit complex fracture behavior. Consequently, a full understanding of this complex behavior is essential.

Consider a unidirectional ceramic matrix composite under a tensile load applied in the direction of the fibers. If no cracking has taken place, the loading results in equal axial strains in the matrix and the fiber. In some ceramic matrix composites, the fracture strain of the fiber is much higher than that of the matrix. Hence, prefailure damage under a tensile load may involve extensive cracks in the matrix which are oriented in the matrix perpendicular to the loading. A few examples of materials that exhibit such a behavior include glass ceramics reinforced by carbon (Brennan and Prewo, 1982) and silicon carbide fibers (Marshall and Evans, 1985). In many cases where the fiber is strong and the interfaces are weak, these cracks are more or less of equal length and are equally spaced in the matrix (Marshall *et al.*, 1985). These cracks are of major concern as they signify the onset of permanent damage and (or) catastrophic failure. Also, since many practical applications involve a corrosive atmosphere, the protection provided by the matrix for the fibers against corrosion can also be lost. These concerns make understanding the mechanics of matrix fracture in ceramic matrix composites important.

The pioneering work on matrix cracking in brittle matrix fiber-reinforced composites has been done by Aveston *et al.* (1971), commonly called the ACK theory. The ACK theory has also been extended and improved upon by Aveston and Kelly (1973), Marshall and Evans (1985), Budiansky *et al.* (1986) and McCartney (1987). These theories have given a better understanding of the strength and toughness of ceramic matrix composites.

Elasticity solutions to a few fracture problems with parallel periodic cracks have been reported in the literature. The problem of a half-plane with an infinite row of periodic cracks was solved by Benthem and Koiter (1973) using an asymptotic expansion to solve the problem. Bowie (1973) used conformal mapping to solve the same problem. The first solution in Cauchy singular integral equation form for interacting arrays of parallel edge cracks was given by Nemat-Nasser *et al.* (1978). Recently, Nied (1987) found an elasticity

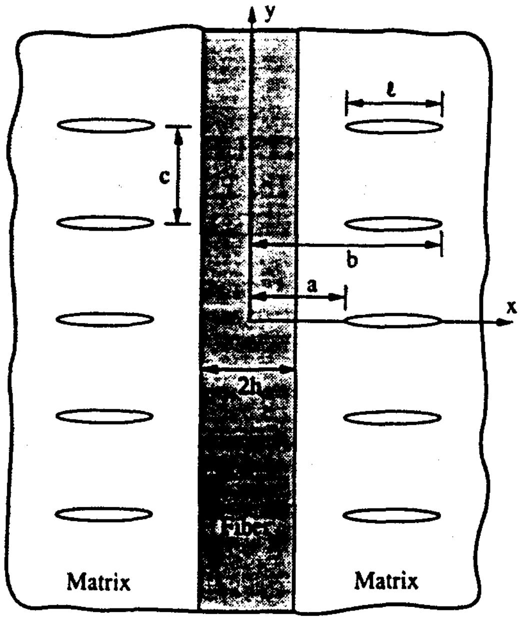

Fig. 1. Schematic of a parallel periodic array of matrix cracks in fiber-reinforced composites.

solution for interacting embedded or edge cracks in a half-plane under uniaxial tension. The solution is given in a strongly singular integral equation form. It should be noted that the results for single cracks or periodic noninteracting cracks are not relevant to understanding the problem of interacting periodic cracks. This is because the stress intensity factors at the crack tips and the crack opening displacements are found to decrease as the spacing between the cracks is reduced.

In this paper, the problem of a fiber embedded in a matrix with parallel periodically spaced cracks is solved (Fig. 1). An arbitrary normal load is applied perpendicular to the cracks. An elasticity solution is found in terms of a hypersingular integral equation. The stress intensity factors at the crack tips and the maximum crack opening displacements are found numerically and evaluated as a function of fiber and matrix moduli, and length, spacing and location of cracks in the matrix.

FORMULATION

The geometry of the problem, shown in Fig. 1, consists of a fiber approximated by an infinite, isotropic, linearly elastic strip with shear modulus, μ_1, Poisson's ratio, ν_1 and width, $2h$. The fiber is perfectly bonded to a matrix approximated by two isotropic, linearly-elastic half-planes with shear modulus, μ_2 and Poisson's ratio, ν_2. The matrix is assumed to have a large number of small cracks of equal length, $b-a$, spaced equally and periodically at a distance c from each other. The crack starts at a distance of $a-h$ from the fiber-matrix interface.

The displacement field for an infinite strip (fiber) is given by (Sneddon and Lowengrub, 1969):

$$u_1(x, y) = -\frac{2}{\pi}\int_0^\infty \left\{\frac{1}{s}\left[f_1(s) - \frac{\kappa_1 - 1}{2} g_1(s)\right]\sinh(sx) + x g_1(s)\cosh(sx)\right\}\cos(sy)\,ds, \tag{1a}$$

$$v_1(x, y) = \frac{2}{\pi}\int_0^\infty \left\{\frac{1}{s}\left[f_1(s) + \frac{\kappa_1 + 1}{2} g_1(s)\right]\cosh(sx) + x g_1(s)\sinh(sx)\right\}\sin(sy)\,ds, \tag{1b}$$

where

$\kappa_1 = (3-\nu_1)/(1+\nu_1)$ for generalized plane stress,

$\kappa_1 = 3-4\nu_1$ for plane strain.

The corresponding stress fields are given by

$$\frac{\sigma^1_{xx}(x,y)}{2\mu_1} = -\frac{2}{\pi}\int_0^\infty [f_1(s)\cosh(sx)+sxg_1(s)\sinh(sx)]\cos(sy)\,ds, \tag{2a}$$

$$\frac{\sigma^1_{yy}(x,y)}{2\mu_1} = \frac{2}{\pi}\int_0^\infty \{[f_1(s)+2g_1(s)]\cosh(sx)+sxg_1(s)\sinh(sx)\}\cos(sy)\,ds, \tag{2b}$$

$$\frac{\sigma^1_{xy}(x,y)}{2\mu_1} = \frac{2}{\pi}\int_0^\infty \{[f_1(s)+g_1(s)]\sinh(sx)+sxg_1(s)\cosh(sx)\}\sin(sy)\,ds. \tag{2c}$$

The displacement field for a half-plane (matrix) with a single crack at $y = 0$ is given by (Sneddon and Lowengrub, 1969):

$$u_2(x,y) = -\frac{2}{\pi}\int_0^\infty \left\{\frac{1}{s}\left[f_2(s)+\frac{\kappa_2-1}{2}g_2(s)\right]+xg_2(s)\right\}e^{-sx}\cos(sy)\,ds$$
$$-\frac{2}{\pi}\int_0^\infty \frac{m(r)}{r}\left(\frac{\kappa_2-1}{2}-ry\right)e^{-ry}\sin(rx)\,dr, \tag{3a}$$

$$v_2(x,y) = -\frac{2}{\pi}\int_0^\infty \left\{\frac{1}{s}\left[f_2(s)-\frac{\kappa_2+1}{2}g_2(s)\right]+xg_2(s)\right\}e^{-sx}\sin(sy)\,ds$$
$$+\frac{2}{\pi}\int_0^\infty \frac{m(r)}{r}\left(\frac{\kappa_2+1}{2}+ry\right)e^{-ry}\cos(rx)\,dr, \tag{3b}$$

where

$\kappa_2 = (3-\nu_2)/(1+\nu_2)$ for generalized plane stress,

$\kappa_2 = 3-4\nu_2$ for plane strain.

The corresponding stress field is given by

$$\frac{\sigma^2_{xx}(x,y)}{2\mu_2} = -\frac{2}{\pi}\int_0^\infty [f_2(s)+sxg_2(s)]e^{-sx}\cos(sy)\,ds-\frac{2}{\pi}\int_0^\infty m(r)(1-ry)e^{-ry}\cos(rx)\,dr, \tag{4a}$$

$$\frac{\sigma^2_{yy}(x,y)}{2\mu_2} = \frac{2}{\pi}\int_0^\infty [f_2(s)+(sx-2)g_2(s)]e^{-sx}\cos(sy)\,ds-\frac{2}{\pi}\int_0^\infty m(r)(1+ry)e^{-ry}\cos(rx)\,dr, \tag{4b}$$

$$\frac{\sigma^2_{xy}(x,y)}{2\mu_2} = \frac{2}{\pi}\int_0^\infty [f_2(s)+(sx-1)g_2(s)]e^{-sx}\sin(sy)\,ds-\frac{2}{\pi}\int_0^\infty m(r)rye^{-ry}\sin(rx)\,dr. \tag{4c}$$

The functions f_1, f_2, g_1, g_2 and m are the unknown functions in the domain $[0,\infty)$ in the stress-displacement fields [eqns (1)–(4)] of the fiber and the matrix, and are found by using the boundary, continuity and symmetry conditions of the problem.

The following are the boundary, continuity and symmetry conditions of the perturbation problem:

$$\sigma_{xx}^{1}(h, y) = \sigma_{xx}^{2}(h, y), \quad 0 \leqslant |y| < \infty, \tag{5a}$$

$$\sigma_{xy}^{1}(h, y) = \sigma_{xy}^{2}(h, y), \quad 0 \leqslant |y| < \infty, \tag{5b}$$

$$v_1(h, y) = v_2(h, y), \quad 0 \leqslant |y| < \infty, \tag{5c}$$

$$u_1(h, y) = u_2(h, y), \quad 0 \leqslant |y| < \infty, \tag{5d}$$

$$\sigma_{xy}^{1}(x, nc) = 0, \quad |x| < h, \quad n = -\infty, \ldots, \infty, \tag{5e}$$

$$\sigma_{xy}^{2}(x, nc) = 0, \quad h < |x| < \infty, \quad n = -\infty, \ldots, \infty, \tag{5f}$$

$$v_1(x, nc) = 0, \quad 0 < |x| < h, \quad n = -\infty, \ldots, \infty, \tag{5g}$$

$$v_2(x, nc) = 0, \quad h < |x| < a, \quad b < |x| < \infty, \quad n = -\infty, \ldots, \infty, \tag{5h}$$

$$\sigma_{yy}^{2}(x, nc) = -p(x), \quad a < |x| < b, \quad n = -\infty, \ldots, \infty. \tag{5i}$$

Equations (5a–d) are the stress and displacement continuity conditions at the interface. Equations (5e–f) are the symmetry conditions. Equations (5g–i) are the mixed boundary conditions on the line of the crack. The pressure $p(x)$ is the arbitrary traction on each of the cracks.

The problem of periodic cracks is solved by first finding the normal stress in the y-direction at any x, y location for the problem of a single pressurized crack at $y = 0$ (or $n = 0$). The results from this problem are then superimposed by shifting the x-axis to all the infinite crack locations in the solution of the single crack. This gives the solution to the crack problem of the infinite periodic cracks of Fig. 1.

For the problem of a single crack, the symmetry conditions (5e–f) are identically satisfied by the shear stress expressions (2c) and (4c). The unknown functions f_1, g_1, f_2, g_2 in the stress-displacement equations (1)–(4) are determined by using the four continuity conditions at the interface given in eqns (5a–d) in terms of integrals of the unknown function m.

For $y = 0$, eqn (3b) gives

$$v(x) \equiv v_2(x, 0) = \frac{2}{\pi} \int_0^{\infty} \frac{m(r)}{r} \left(\frac{\kappa_2 + 1}{2} \right) \cos(rx) \, dr. \tag{6}$$

Inverting eqn (6):

$$\frac{m(r)}{r} = \frac{2}{\kappa_2 + 1} \int_a^b v(t) \cos(rt) \, dt. \tag{7}$$

Using the results of eqn (7), the unknowns f_1, g_1, f_2, g_2 of eqns (1)–(4) can now be written in terms of the integrals of $v(t)$, $a < t < b$, the half-crack opening displacement in the matrix. Substituting these values in eqn (4b), the normal stress in the y-direction in the matrix due to a single crack can be written as

$$\frac{\pi(1 + \kappa_2)\sigma_{yy}^{2}(x, y)}{4\mu_2} = \int_a^b v(t)[K_a(x, t, y) + K_b(x, t, y)] \, dt, \tag{8}$$

where

$$K_a(x, t, y) = \frac{3y^4 - 6(t - x)^2 y^2 - (t - x)^4}{[y^2 + (t - x)^2]^3}, \tag{9a}$$

$$K_b(x, t, y) = \int_0^{\infty} k_b(x, t, s) \cos(sy) \, ds, \tag{9b}$$

$$k_b(x,t,s) = e^{-s(x+t-2h)} \sum_{i=1}^{3} \sum_{j=2}^{5} \frac{B_{ij} s^{j-1} e^{-2sh(i-1)}}{1+Z(s)}, \tag{9c}$$

$$\begin{aligned}
B_{12} &= -B_{32} = -9q_2^2 - q_1q_2 - 9q_2 - q_1, \\
B_{13} &= -B_{33} = 6q_2^2x + 6q_2x + 6q_2^2t + 6q_2t - 12hq_2^2 - 12hq_2, \\
B_{14} &= -B_{34} = -4q_2^2tx - 4q_2tx + 4hq_2^2x + 4hq_2x + 4hq_2^2t + 4hq_2t - 4h^2q_2^2 - 4h^2q_2, \\
B_{15} &= B_{35} = 0, \\
B_{22} &= -6q_1q_2^2 - 6q_1q_2 + 6q_2 + 6, \\
B_{23} &= 2q_1q_2^2x + 2q_1q_2x - 2q_2x - 2x + 2q_1q_2^2t + 2q_1q_2t - 2q_2t - 2t \\
&\qquad - 40hq_1q_2^2 - 36hq_2^2 - 4hq_1q_2 + 4hq_2 - 4hq_1, \\
B_{24} &= 24hq_1q_2^2x + 24hq_2^2x + 24hq_1q_2^2t + 24hq_2^2t - 48h^2q_1q_2^2 - 48h^2q_2^2, \\
B_{25} &= -16hq_1q_2^2tx - 16hq_2^2tx + 16h^2q_1q_2^2x + 16h^2q_2^2x + 16h^2q_1q_2^2t \\
&\qquad + 16h^2q_2^2t - 16h^3q_1q_2^2 - 16h^3q_2^2,
\end{aligned} \tag{10}$$

$$q_1 = \frac{\kappa_2\mu_1 - \kappa_1\mu_2}{\mu_2 + \kappa_2\mu_1}, \tag{11a}$$

$$q_2 = \frac{\mu_2 - \mu_1}{\mu_1 + \kappa_1\mu_2}, \tag{11b}$$

$$Z(s) = a_1 s e^{-2sh} + a_2 e^{-4sh}, \tag{12a}$$

$$a_1 = 4hq_2 \frac{1+q_1}{1+q_2}, \tag{12b}$$

$$a_2 = -q_1q_2. \tag{12c}$$

Note that B_{ij}'s and $Z(s)$ are expressed in terms of only two independent material parameters q_1 and q_2 (Dundurs, 1967) instead of the four material constants of the constituents, namely μ_1, μ_2, κ_1 and κ_2.

In eqn (8), the function $K_a(x, t, y)$ is known explicitly as given by eqn (9a), while $K_b(x, t, y)$ is known in terms of a semi-infinite integral as given by eqn (9b). One needs to find the function $K_b(x, t, y)$ explicitly so that the infinite summations required later to solve the problem of periodic cracks can also be carried out explicitly. To obtain the function $K_b(x, t, y)$ explictly, the function $1/[1+Z(s)]$ in eqn (9c) can be approximated by a Maclaurin series:

$$\frac{1}{1+Z(s)} \approx 1 + \sum_{i=1}^{N} [-Z(s)]^i, \quad |Z(s)| < 1, \quad 0 < s < \infty. \tag{13}$$

This expansion can be substituted in eqn (9c) to evaluate $K_b(x, t, y)$ explicitly. The question remains of how many terms in the Maclaurin series are required for an accurate representation of equation (9b)? If ε_s is the prespecified tolerance within which the function $1/[1+Z(s)]$ is to be approximated for any value of s, stiffness properties, and geometry, then the number of terms required in the series is given by finding the value of N when

$$\frac{|F_e(s) - F_a(s)|}{|F_e(s)|} < \varepsilon_s, \tag{14a}$$

$$F_e(s) = \left| \frac{1}{1+Z_{max}(s)} \right|. \tag{14b}$$

$$F_a(s) = \left|1+\sum_{i=1}^{N}[-Z_{max}(s)]^i\right|, \tag{14c}$$

where $Z_{max}(s)$ is the maximum value of $|Z(s)|$ found by

$$Z_{max}(s) = \text{maximum}\,[|Z(0)|, \quad |Z(s_{max})|], \tag{15}$$

and s_{max} is the root of the equation $Z'(s) = 0$.

Since the range of the independent constants is $-1 < q_1 < 3$ and $-1 < q_2 < 1$, the maximum value of $|Z(s)|$, from eqn (12), is one. In such a case, the Maclaurin series expansion (13) could be divergent. However, most ceramic matrix composites fall within the range of the ratio of the stiffness properties of the matrix and the fiber as $1/6 < \mu_f/\mu_m < 6$ range. For this range, the maximum value of $|Z(s)|$ is 85/133. For such values only a finite number of terms are required in the Maclaurin series expansion (13). For example, to yield an absolute relative true error of less than 1×10^{-8} in the series expansion (13), one requires at least 41 terms in the Maclaurin series. Note that this is the number of terms required for the accuracy to be at least 1×10^{-8} of the function $1/[1+Z(s)]$ for any value of s. At points other than where the maximum value of $|Z(s)|$ occurs, the accuracy of the Maclaurin series representation (13) is still higher. Also note that the number of terms in the series expansion (13) are found automatically in the numerical program for this problem based on the prescribed error tolerance and material properties.

Substituting eqn (12a) in eqn (13), the function $1/[1+Z(s)]$ can now be expressed as

$$\frac{1}{1+Z(s)} \approx 1+\sum_{i=1}^{M}(-1)^i[a_1 s\,e^{-2sh}+a_2\,e^{-4sh}]^i$$

$$= 1+\sum_{k=1}^{M}\sum_{l=1}^{k+1} D_{kl}s^{k-l+1}\,e^{-2sh(k+l-1)}, \tag{16}$$

where

$$D_{kl} = (-1)^k a_1^{k-l+1}a_2^{l-1}\frac{k!}{(k-l+1)!(l-1)!}. \tag{17}$$

Substituting eqns (16) and (17) in eqn (9c), and regrouping the powers of s and the exponential terms, one obtains

$$k_b(x,t,s) = \sum_{i=1}^{2M+3}\sum_{j=1}^{M+5} C_{ij}s^{j-1}\,e^{-2sh(i-1)-s(x+t-2h)}, \tag{18}$$

where C_{ij}'s are found by adding $B_{ij}D_{kl}$ as the contribution to the $C_{k+l+i-1,k-l+j+1}$ term.

The kernel $K_b(x,t,y)$ can now be found explicitly by substituting eqn (18) in eqn (9b) and integrating analytically to give

$$K_b(x,t,y) = \sum_{i=1}^{2M+3}\sum_{j=1}^{M+5} C_{ij}(-1)^{j-1}\frac{d^{j-1}}{dx^{j-1}}\left\{\frac{2h(i-1)+(x+t-2h)}{[2h(i-1)+(x+t-2h)]^2+y^2}\right\}. \tag{19}$$

Substituting eqns (9a) and (19) in eqn (8), one finally obtains an expression for the matrix normal stress where the kernels are known explicitly; it is given by

$$\frac{\pi(1+\kappa_2)\sigma_{yy}^2(x,y)}{4\mu_2} = \int_a^b v(t)\left[\frac{3y^4-6(t-x)^2y^2-(t-x)^4}{[y^2+(t-x)^2]^3}\right.$$
$$\left.+\sum_{i=1}^{2M+3}\sum_{j=1}^{M+5} C_{ij}(-1)^{j-1}\frac{\mathrm{d}^{j-1}}{\mathrm{d}x^{j-1}}\left\{\frac{2h(i-1)+(x+t-2h)}{[2h(i-1)+(x+t-2h)]^2+y^2}\right\}\right]\mathrm{d}t. \quad (20)$$

Now one can find the expression for the matrix normal stress at $y = 0$ for the infinite periodic cracks by superposition as follows (Nied, 1987). The change in the normal stress in the y-direction in the matrix along $y = 0$ due to the infinite periodic cracks is the infinite sum of contributions from eqn (20) for values of $y = nc$, $n = -\infty,\ldots,\infty$. Putting $y = nc$ in eqn (20) and conducting the summation on n from $-\infty$ to ∞, one obtains the singular integral equation for the problem of an infinite array of periodic cracks as

$$-\frac{\pi(1+\kappa_2)p(x)}{4\mu_2} = \int_a^b \frac{v(t)}{(t-x)^2}\mathrm{d}t+\int_a^b v(t)[K_a^*(x,t)+K_b^*(x,t)]\,\mathrm{d}t, \quad a<x<b, \quad (21)$$

where

$$K_a^*(x,t) = -\frac{6}{c^2}h_1\left\{\frac{t-x}{c}\right\}-12\frac{(t-x)^2}{c^6}h_2\left\{\frac{t-x}{c}\right\}-\frac{(t-x)^4}{c^6}h_3\left\{\frac{t-x}{c}\right\}, \quad (22a)$$

$$K_b^*(x,t) = \sum_{i=1}^{2M+3}\sum_{j=1}^{M+5} C_{ij}(-1)^{j-1}\frac{\pi}{c}\frac{\mathrm{d}^{j-1}}{\mathrm{d}x^{j-1}}\left\{\coth\left[\frac{\pi\{2h(i-1)+(x+t-2h)\}}{c}\right]\right\}, \quad (22b)$$

$$h_1(x) = \frac{1}{16}[xw_2'(x)+3x^2w_1''(x)]-w_2(x), \quad (22c)$$

$$h_2(x) = \frac{1}{16}\left[w_1''(x)+3\frac{w_2'(x)}{x}\right], \quad (22d)$$

$$h_3(x) = \frac{1}{16}\left[\frac{w_2'(x)}{x^3}+3\frac{w_1''(x)}{x^2}\right], \quad (22e)$$

$$w_1(x) = \sum_{n=1}^{\infty}\frac{1}{x^2+n^2} = \frac{1}{2x^2}[\pi x\coth(\pi x)-1](=\pi^2/6, \ \text{if } x=0). \quad (22f)$$

$$w_2(x) = \sum_{n=1}^{\infty}\frac{x^2-n^2}{(x^2+n^2)^2} = \frac{\pi^2}{2}\operatorname{cosech}^2(\pi x)-\frac{1}{2x^2}(=-\pi^2/6, \ \text{if } x=0). \quad (22g)$$

$$w_3(x) = \sum_{n=-\infty}^{\infty}\frac{x}{x^2+n^2} = \pi\coth(\pi x). \quad (22h)$$

Equation (21) has $1/(t-x)^2$ integrands which are called strong singularities. Such singularities are classically non-integrable and cannot be defined even in the Cauchy principal value sense. However, such problems can be solved provided the integral is interpreted in the Hadamard (1923) sense by retaining the finite part only. This concept was used by Kaya and Erdogan (1987) to develop formulae for strong singular integrals as found in this problem. The derivatives of the coth function in eqn (22b) were found by using the MACSYMA (1985) symbolic manipulator.

SOLUTION OF INTEGRAL EQUATIONS

Using the following non-dimensional parameters:

$$t = \frac{(b-a)}{2}\tau + \frac{(b+a)}{2}, \tag{23a}$$

$$x = \frac{(b-a)}{2}\xi + \frac{(b+a)}{2}, \tag{23b}$$

$$v(t) = \frac{(b-a)}{2}V(\tau), \tag{23c}$$

$$p(x) = P(\xi), \tag{23d}$$

eqn (21) can be normalized over the range $[-1, 1]$ as

$$-\frac{\pi(1+\kappa_2)P(\xi)}{4\mu_2} = \int_{-1}^{1} \frac{V(\tau)}{(\tau-\xi)^2}\,d\tau + \int_{-1}^{1} V(\tau)M(\tau,\xi)\,d\tau, \quad -1 < \xi < 1, \tag{24}$$

where

$$M(\tau,\xi) = \left(\frac{b-a}{2}\right)^2 [K_a^*(x,t) + K_b^*(x,t)], \quad -1 < \xi < 1. \tag{25}$$

The singular integral eqn (24) is solved numerically by assuming the displacement function of the form:

$$V(\tau) = \sum_{i=0}^{N} A_i U_i(\tau)(1-\tau)^{1/2}(1+\tau)^{\gamma}, \quad -1 < \tau < 1, \tag{26}$$

where $U_i(\tau)$ is the Chebyshev polynomial of the second kind, and A_i, $i = 0, 1, 2, \ldots, N$, are the unknown constants. The form of the displacement given by eqn (26) preserves the singularity of the slope of the crack opening displacement. The value of γ is one-half if the crack is internal in the matrix, while γ is given by the root of the characteristic equation

$$2\cos\pi(1-\gamma) + 4q_2\gamma^2 - (q_1+q_2) = 0, \tag{27}$$

if the crack touches the interface. Equation (27) can be derived by conducting an asymptotic analysis on the corresponding flux function integral equation of eqn (8) by following asymptotic analysis techniques (Gupta, 1973; Kaw and Goree, 1991).

Substituting eqn (26) in eqn (24), one obtains

$$-\frac{\pi(1+\kappa_2)P(\xi)}{4\mu_2} = \sum_{i=0}^{N} A_i\left[\int_{-1}^{1} \frac{U_i(\tau)(1-\tau)^{1/2}(1+\tau)^{\gamma}}{(\tau-\xi)^2}\,d\tau + \int_{-1}^{1} U_i(\tau)(1-\tau)^{1/2}(1+\tau)^{\gamma}M(\tau,\xi)\,d\tau\right], \quad -1 < \xi < 1. \tag{28}$$

The first integral in eqn (28) can be evaluated exactly in the Hadamard sense (1923) for $\gamma = 1/2$ as

$$\oint_{-1}^{1} \frac{U_n(\tau)\sqrt{1-\tau^2}}{(\tau-\xi)^2} \mathrm{d}\tau = -\pi(n+1)U_n(\xi), \quad -1 < \xi < 1. \tag{29}$$

For $\gamma \neq 1/2$, the first integral in eqn (28) is solved by one of the techniques given in Kaw (1991a, b), where the function

$$F_n(\xi) = \oint_{-1}^{1} \frac{U_n(\tau)(1-\tau)^{1/2}(1+\tau)^{\gamma}}{(\tau-\xi)} \mathrm{d}\tau, \quad -1 < \xi < 1, \tag{30}$$

is evaluated numerically in the Cauchy Principal value sense and then the first integral in eqn (28) is found by differentiating the function $F_n(\xi)$ numerically to get

$$\oint_{-1}^{1} \frac{U_n(\tau)(1-\tau)^{1/2}(1+\tau)^{\gamma}}{(\tau-\xi)^2} d\tau = \frac{\mathrm{d}F_n(\xi)}{\mathrm{d}\xi}, \quad -1 < \xi < 1. \tag{31}$$

In the present paper, $F_n(\xi)$ was evaluated by using the IMSL (1987) DQDAWC routine which adaptively finds the integral of the form

$$\oint_{a}^{b} f(x)/(x-c)\, \mathrm{d}x, \quad a < c < b,$$

in Cauchy Principal value sense. The derivative of the function $F_n(\xi)$ is then evaluated using the IMSL (1987) DDERIV routine, which finds the derivative of a function by adaptively changing the step size to enhance accuracy. The second integral in eqn (28) is solved by using the IMSL (1987) adaptive integration routine DQDAWS.

A collocation method is used to solve for the unknown coefficients, A_i. $(N+1)$ collocation points, ξ_i, $i = 0, 1, \ldots, N$, distributed on the interval $(-1, 1)$ with more points near the ends of the interval, are chosen to result in a set of $(N+1)$ linear algebraic equations to give

$$-\frac{\pi(1+\kappa_2)P(\xi_i)}{4\mu_2} = \sum_{i=0}^{N} A_i \left[\oint_{-1}^{1} \frac{U_i(\tau)(1-\tau)^{1/2}(1+\tau)^{\gamma}}{(\tau-\xi_i)^2} \mathrm{d}\tau + \int_{-1}^{1} U_i(\tau)(1-\tau)^{1/2}(1+\tau)^{\gamma} M(\tau, \xi_i)\, \mathrm{d}\tau \right], \quad i = 0, 1, 2, \ldots, N. \tag{32}$$

The roots ξ_i are given by

$$\xi_i = \cos\left[\frac{\pi(2i+1)}{2(N+1)}\right], \quad i = 0, 1, \ldots, N. \tag{33}$$

For the case of a crack up to the interface ($a = h$), the system of equations becomes ill conditioned. To avoid this, an algorithm based on keeping the number of terms in the expansion series low and increasing the number of collocation points was employed. This resulted in a set of equations where there were more equations than unknowns. This set of equations was solved in the least squares sense by using the IMSL (1987) subroutine DLSQRR. This method has been successfully employed in several other problems in fracture mechanics of solids (Nied, 1987; Kaw and Das, 1991; Kaw and Besterfield, 1991).

The stress intensity factor at $x = b$ for uniform pressure $p(x) = p_0$ at the crack tip away from the interface is

$$K_1(b) = \lim_{x\to b^+} \sqrt{2(x-b)}\sigma_{yy}^2(x,0) = \lim_{x\to b^-} -\frac{4\mu_2}{(\kappa_2+1)}\sqrt{2(b-x)}\frac{dv(x)}{dx}, \tag{34}$$

which can be evaluated numerically from eqns (26) and (23c) as

$$\frac{K_1(b)}{p_0\sqrt{l_0}} = \frac{2^{\gamma+3/2}\mu_2}{(\kappa_2+1)}\sum_{i=0}^{N} A_i(i+1),$$
$$2l_0 = l = (b-a). \tag{35}$$

The stress intensity factor at $x = a$ for uniform pressure $p(x) = p_0$ at the crack tip close to the interface is

$$K_1(a) = \lim_{x\to a^-} \sqrt{2(a-x)}\sigma_{yy}^2(x,0) = 2\mu^*\sqrt{2(x-a)}\lim_{x\to a^+}\frac{dv(x)}{dx}, \quad a > h,$$
$$= \lim_{x\to a^-} \sqrt{2}(a-x)^{1-\gamma}\sigma_{yy}^1(x,0) = 2\mu^*\sqrt{2}(x-a)^{1-\gamma}\lim_{x\to a^+}\frac{dv(x)}{dx}, \quad a = h, \tag{36}$$

which can be evaluated numerically from eqns (26) and (23c) as

$$\frac{K_1(a)}{p_0 l_0^{(1-\gamma)}} = 4\mu^*\gamma\sum_{i=0}^{N} A_i(i+1)(-1)^i, \tag{37}$$

$$\mu^* = \frac{2\mu_2}{(\kappa_2+1)} \quad \text{and} \quad \gamma = 1/2, \quad \text{if } a > h,$$
$$= \frac{\mu_1\mu_2}{\sin\pi(1-\gamma)}\left[\frac{1+2q_1\gamma}{\mu_2+\kappa_2\mu_1} + \frac{1-2q_2\gamma}{\mu_1+\kappa_1\mu_2}\right] \quad \text{if } a = h. \tag{38}$$

RESULTS AND DISCUSSION

Two parameters—the normalized stress intensity factors at the crack tips and maximum normalized crack opening displacements are studied as a function of fiber–matrix moduli ratio, and length, location and spacing of cracks. The results are presented for plane strain with constant pressure on the crack surfaces. Five cases of fiber matrix properties in the range $1/6 < \mu_f/\mu_m < 6$ ($\nu_f = \nu_m = 0.3$) are used in the results.

The stress intensity factors [eqns (35) and (37)] at the crack tips for the periodic cracking problem solved here are normalized by the stress intensity factor for the problem of periodic cracking in a homogeneous infinite plane ($a = \infty$). The normalized stress intensity factors are then a direct measure of the degree of fracture toughening of the composite due to reinforcement. Values of the normalized stress intensity factor close to one show the insensitivity to the fiber reinforcement, while values away from one show the dominance of the fiber reinforcement over crack spacing.

Figures 2 and 3 show the normalized stress intensity factor at $x = a$ and $x = b$, respectively, as a function of the normalized spacing parameter ($l/\{l+c\}$) for constant location of the crack with respect to the interface. The value of the stress intensity factor for the case of periodic cracking in a homogeneous infinite plane ($a = \infty$) ($K_{A\infty} = K_1(a)/\{p_0\sqrt{l_0}\}$) is also plotted, if one is interested in the actual value of the stress intensity factors. The case of $[l/\{2(a-h)\} = 1]$ is close to the values obtained for the limiting case of $a = \infty$. This implies that the fiber reinforcement has no effect on the stress intensity factors when the crack lengths are of the same order or less of the distance from the interface. The same conclusions were drawn by Nied (1987) for the problem of a half-plane with periodic cracks and traction-free surface ($\mu_1 = 0$). It is also seen that the stress intensity factors away from the interface are insensitive to the fiber reinforcement for all crack spacings

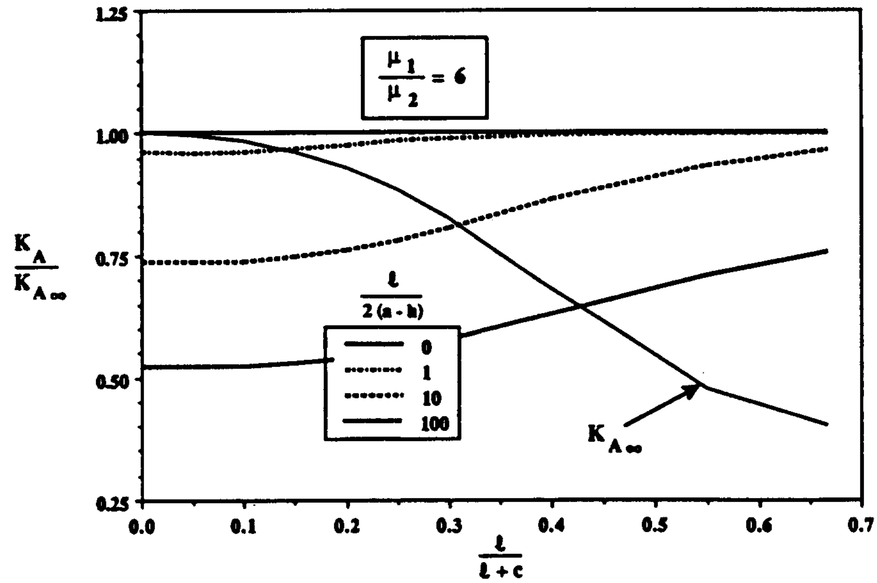

Fig. 2. Normalized stress intensity factor at the crack tip near the interface as a function of normalized crack length for constant crack location.

and crack locations. The stress intensity factor at $x = a$ decreases as the location of the crack comes closer to the interface. This shows that the crack growth towards the interface is stable if the fiber is stiffer than the matrix.

Figures 4 and 5 show the stress intensity factors at $x = a$ and $x = b$, respectively as a function of the crack spacing for constant fiber–matrix moduli ratio. The stress intensity factor at $x = a$ is shown to decrease as a function of the fiber–matrix moduli ratio. Also, the effect of the spacing of the cracks predominates the effect of the fiber–matrix moduli ratio as the crack spacing decreases to the order of the crack length. The stress intensity factor at $x = b$ shows the same effect but not to such a large extent. The effect of the crack spacing in that case predominates that of the moduli ratio at crack spacings as high as one order higher than the crack length.

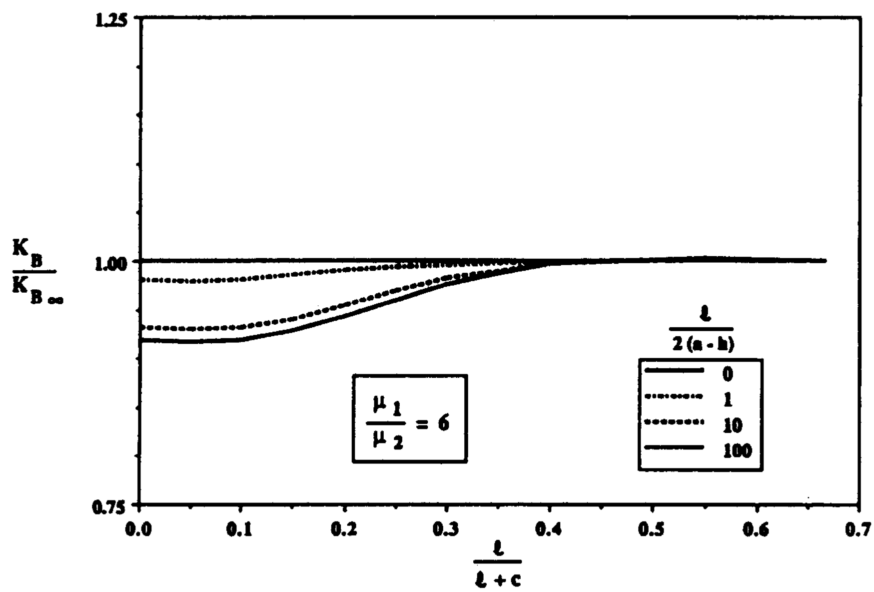

Fig. 3. Normalized stress intensity factor at the crack tip away from the interface as a function of normalized crack length for constant crack location.

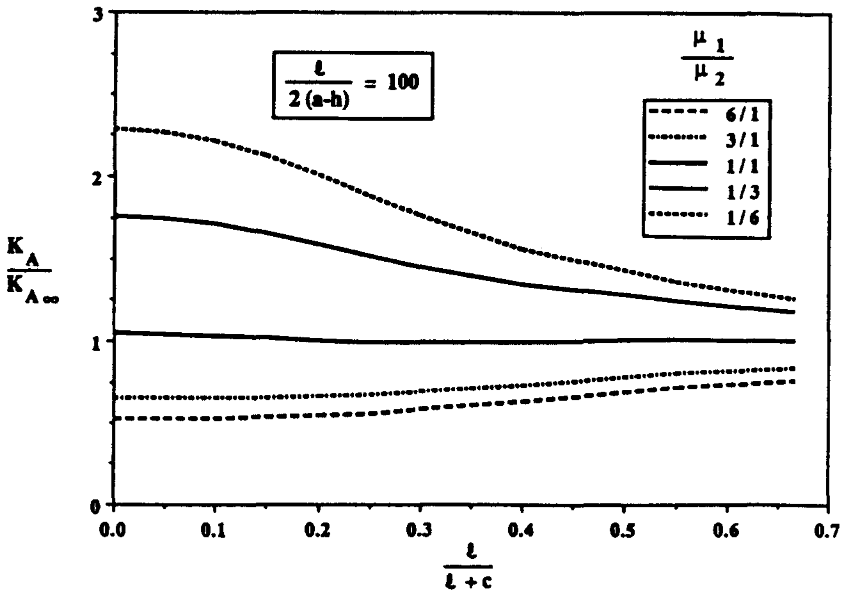

Fig. 4. Normalized stress intensity factor at the crack tip near the interface as a function of normalized crack length for constant fiber-matrix moduli ratio.

The maximum crack opening displacement for the problem solved here is normalized with the maximum crack opening displacement for the problem of periodic cracks in a homogeneous medium ($a = \infty$) of the same material as the matrix. The normalized crack opening displacements are then a direct measure of the degree of exposure of the composite to the external environment. Again, values of the normalized crack opening displacement close to one show the insensitivity to the fiber-reinforcement, while values away from one show the dominance of the fiber-reinforcement over crack spacing.

Figure 6 shows the normalized maximum crack opening displacement as a function of the crack spacing parameter for a constant vicinity of the crack tip to the fiber-matrix interface. The case of $[l/\{2(a-h)\} = 1]$ is close to the values obtained for the limiting case of $a = \infty$. This implies that the fiber-reinforcement has no effect on the crack opening

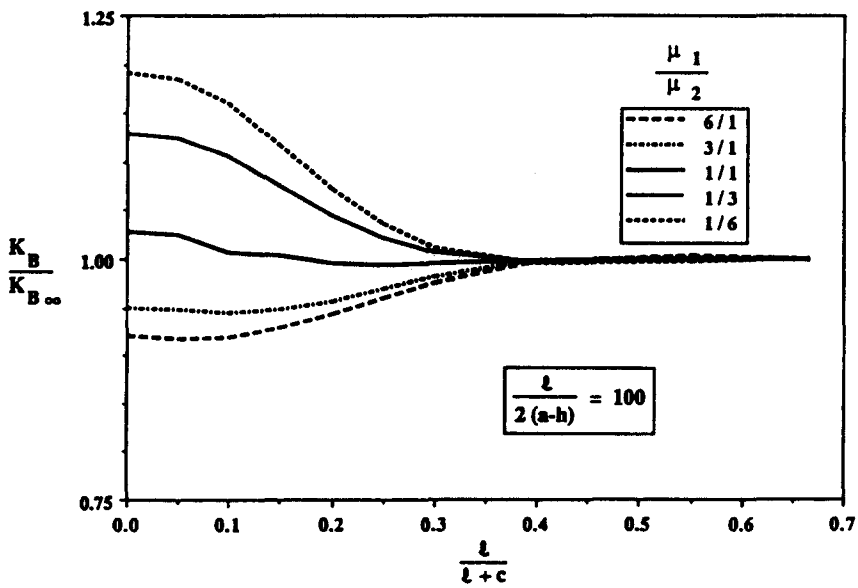

Fig. 5. Normalized stress intensity factor at the crack tip away from the interface as a function of normalized crack length for constant fiber-matrix moduli ratio.

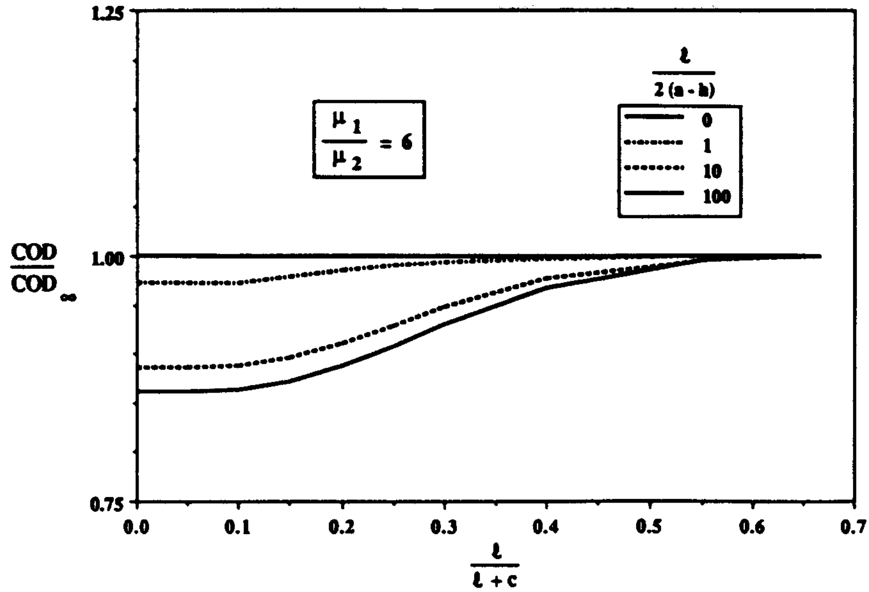

Fig. 6. Normalized crack opening displacement as a function of normalized crack length for constant crack location.

displacement when the crack length is of the same order or less of the distance from the interface. However, as the crack approaches the interface, the fiber reinforcement makes the normalized crack opening displacement smaller. Also, the effect of the crack spacing predominates the effect of the fiber reinforcement, when the crack spacing approaches an order of the crack length.

Figure 7 shows the normalized maximum crack opening displacement as a function of the crack spacing parameter for constant fiber–matrix moduli ratios. The maximum crack opening displacement shows similar behavior as the stress intensity factor at the crack tip away from the interface.

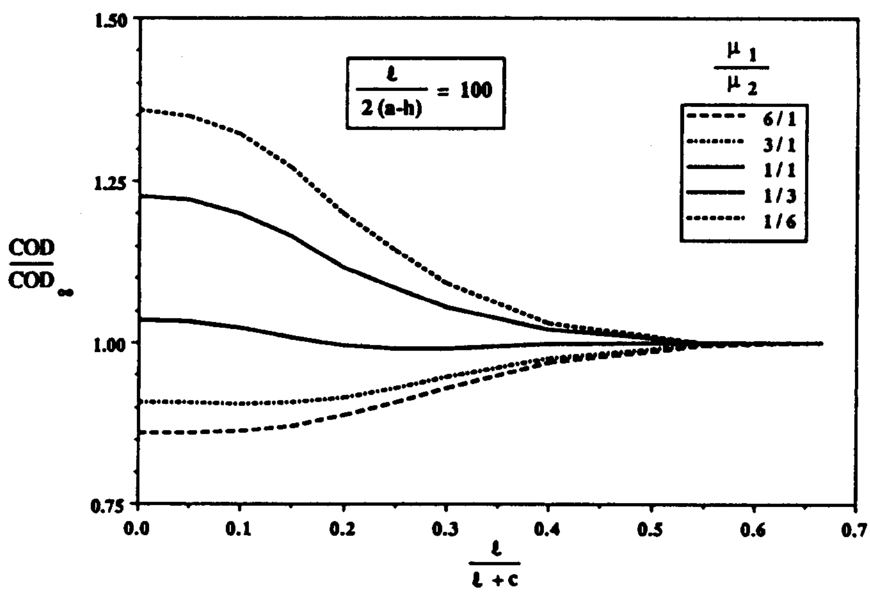

Fig. 7. Normalized crack opening displacement as a function of normalized crack length for constant fiber–matrix moduli ratio.

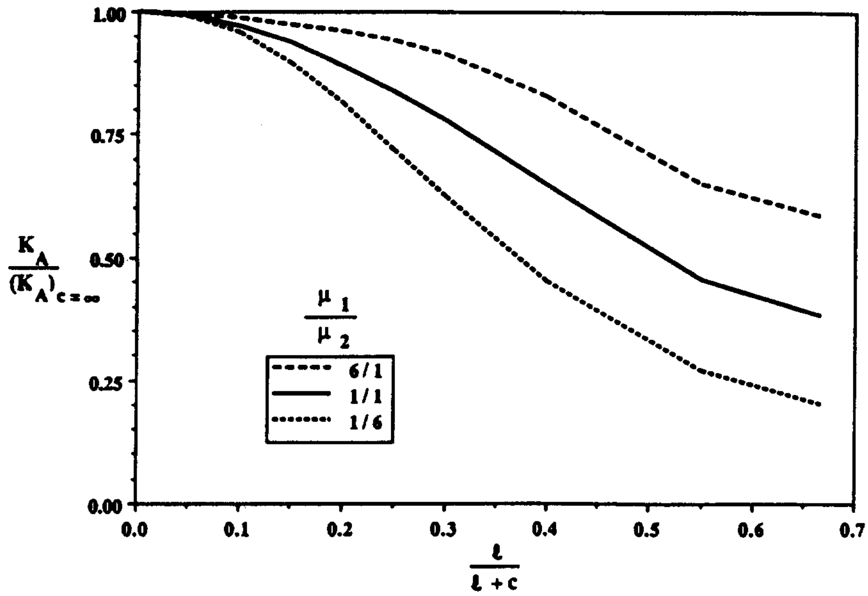

Fig. 8. Normalized stress intensity factor at the crack tip at the interface as a function of normalized crack length for constant fiber–matrix moduli ratio.

The results for the limiting case of the crack touching the interface are shown in Fig. 8. The stress intensity factor at $x = a$ [eqn (37)] cannot be normalized in the same way as the above results [Figs (2)–(5)] because the stress at $x = a$ does not have a square root type singularity. In this case, the stress singularity is given by the root of the characteristic equation (27), while the intensity of the stress singularity is given by eqns (37) and (38). The stress intensity factors in this case are normalized with respect to the corresponding stress intensity factor for a single crack touching the interface ($a = h$, $c = \infty$). From Fig. 7, it shows that the stress intensity factor decreases as a function of the crack spacing, but has a more predominant effect when the fiber is stiffer than the matrix.

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